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RECOVERING A ROTATION MATRIX FROM THREE DIRECTION COSINES

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ABSTRACT

In this paper, we propose a new parameterization method to represent rotation matrices using the angles $\vec{\phi}$ recovered from the three direction cosines that lie on the diagonal. The map from the possible configuration space of the new variable $\vec{\phi}$ to the solid ball model in axis-angle coordinates is constructed. We also introduce a bi-invariant metric and two left-invariant metrics for measuring the distance in configuration space which could be the foundation for path planning in $\vec{\phi}$ space. We further analyze the Jacobian matrix and singularities to better understand the manipulability.

INTRODUCTION

The parameterization of the rotation group $SO(3)$ is a basic topic in the kinematics community that has been illustrated in various coordinate systems, such as Euler angles [1], axis-angle coordinates [2], direction cosines [3], etc. A rotation matrix describes the orientation of a body-fixed right-hand orthogonal frame relative to an inertial right-hand orthogonal frame, and is invariant to the coordinates or representations. Different parameterization methods are superior in different application areas. For comprehensive and classic descriptions of rotations and motions, see [4] [5] [6] [7]. For overviews of parameterization methods, see [8] [9]. For relatively recent rotation parameterizations, see [10] [11] [12].

Axis-angle coordinates for pure rotations have two key elements which are rotation magnitude θ and axis $\vec{n}$. In the kinematics of series manipulators, the forward kinematics between the end effector and the base link can be conveniently expressed as the product of exponentials [13]. The geometric model for the special orthogonal group $SO(3)$ [14] [15] can be generated from θ and $\vec{n}$, which can be used as coordinates for the solid ball of radius π. Each point in the ball corresponds to a rotation.

In spacecraft kinematics, a rotation matrix is usually called a direction cosine matrix. This is because each element in the matrix can be explained as the cosine between a unit axes vector in body-fixed frame and one in spatial-fixed frame [16]. In this paper, instead of using nine direction cosines to fill in the rotation matrix, we recover it using the three direction cosines sitting on the diagonal, the angles recovered from which are denoted as $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$. $\vec{\phi}$ is the new variable vector in our three direction cosine coordinates.

The remainder of this paper is organized as follows. We first review two parameterizations of $SO(3)$: (1) axis-angle coordinates, and (2) direction cosines. The rotation magnitude θ, axis $\vec{n}$ and matrix R are then represented in terms of $\vec{\phi}$. We discuss three cases for there different representation for $\vec{n}$. Another interesting thing we discussed is the $\vec{\phi}$ configuration space. The onto map from $\vec{\phi}$ space to the solid ball as one geometric model for $SO(3)$ is constructed. After getting the rotation matrix, we discuss several properties of it such as the transpose and the inverse of the rotation matrix. Three new metrics in terms of $\vec{\phi}$ are then developed for measuring the magnitude of rotation. To understand the manipulability of the new parameterization, we find the Jacobian matrix and singular configurations. Switching singularities and boundary singularities are further defined based on their different relationships with the Jacobian. The paper concludes with a simple trajectory mapping task from $R(t)$ to $\vec{\phi}(t)$.

PARAMETERIZATION OF A ROTATION

Axis-angle Coordinates

When only considering the pure rotation of a rigid body about a given axis by some amount, the orientation of the body is described as the relative orientation of a coordinate frame attached to the body and a fixed frame. All the coordinate frames are right-handed without extra statement. The body fixed frame is coincident with the spatial fixed frame before rotating.

For axis-angle coordinates, the rotation can be represented as

$$R = \exp(\theta N) \tag{1}$$

where

$$N = \hat{n} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n1 & 0 \end{pmatrix}, \tag{2}$$

and $\vec{n} = (n_1, n_2, n_3)^T$ is the unit vector specifies the rotation direction. θ denotes the rotation magnitude.

An efficient way to calculate $\exp(\theta N)$ is using Rodrigues' formula [17],

$$R = I + \sin\theta N + (1-\cos\theta)N^2. \tag{3}$$

Defining $v(\theta) = 1-\cos(\theta)$ and using $s\theta$ and $c\theta$ to represent $\sin\theta$ and $\cos\theta$ respectively, the rotation matrix R can then be expanded as

$$R = \begin{pmatrix} c\theta + n_1^2 v(\theta) & n_1 n_2 v(\theta) - n_3 s\theta & n_1 w_3 v(\theta) + n_2 s\theta \\ n_1 n_2 v(\theta) + n_3 s\theta & c\theta + n_2^2 v(\theta) & n_2 n_3 v(\theta) - n_1 s\theta \\ n_1 n_3 v(\theta) - n_2 s\theta & n_1 s\theta + n_2 n_3 v(\theta) & c\theta + n_3^2 v(\theta) \end{pmatrix}. \tag{4}$$

Direction Cosine Matrix

As a general way of notation, we have the rotation matrix as

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \tag{5}$$

The matrix stacked by the unit length axes vectors of spatial frame is $S = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$. Without loss of generality, by reordering e_i, we could have $S = I$. The corresponding matrix stacked by unit length axes vectors of the body fixed frame viewed in spatial fixed frame is $T = (\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}')$.

Suppose the body fixed frame is rotated by R. We have

$$\begin{aligned} \left(\vec{e}_1{}' \ \vec{e}_2{}' \ \vec{e}_3{}'\right) &= R\left(\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3\right) \\ R &= \left(\vec{e}_1{}' \ \vec{e}_2{}' \ \vec{e}_3{}'\right)\left(\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3\right)^T \\ &= \begin{pmatrix} \vec{e}_1{}' \cdot \vec{e}_1 & \vec{e}_1{}' \cdot \vec{e}_2 & \vec{e}_1{}' \cdot \vec{e}_3 \\ \vec{e}_2{}' \cdot \vec{e}_1 & \vec{e}_2{}' \cdot \vec{e}_2 & \vec{e}_2{}' \cdot \vec{e}_3 \\ \vec{e}_3{}' \cdot \vec{e}_1 & \vec{e}_3{}' \cdot \vec{e}_2 & \vec{e}_3{}' \cdot \vec{e}_3 \end{pmatrix}. \end{aligned} \tag{6}$$

Equating Eqn.6 with Eqn.5, we have

$$\vec{e}_i{}' \cdot \vec{e}_j = r_{ij} = \|e_i\|\|e_j\| \cos\phi_{ij} = \cos\phi_{ij}. \tag{7}$$

Thus, r_{ij} is the cosine of the angle between e_i' and e_j and that is why the rotation matrix is so called direction cosine matrix.

ROTATION MATRIX BASED ON THREE DIRECTTION COSINES

The definition of the variables for the three direction cosines parameterization is the angles recovered from the diagonal elements of rotation matrix denoted as ϕ_1, ϕ_2, and ϕ_3. A visual demonstration is shown in Fig.1. The $x_0y_0z_0$ frame is the spatial fixed frame and the $x'y'z'$ frame is the body fixed frame.

When $i = j$,

$$\vec{e}_i{}' \cdot \vec{e}_i = \|e_i\|^2 \cos\phi_i = \cos\phi_i = r_{ii}. \tag{8}$$

The trace of rotation matrix in terms of θ and $\vec{\phi}$ is

$$\mathrm{trace}(R) = 1 + 2c\theta = \cos\phi_1 + \cos\phi_2 + \cos\phi_3. \tag{9}$$

The rotation magnitude can then be represented using $\vec{\phi}$ as

$$\theta = \arccos\frac{\cos\phi_1 + \cos\phi_2 + \cos\phi_3 - 1}{2}. \tag{10}$$

Another key element for the axis-angle coordinates is the rotation axis $\vec{n}$. To get the map from $\vec{n}$ to $\vec{\phi}$ and also the rotation matrix, three cases are discussed.

Case I

When $\phi_1 = \phi_2 = \phi_3 = 0$, it is obvious that $R = I$, which is the identity matrix. In this case, the body frame is coincident with the spatial frame.

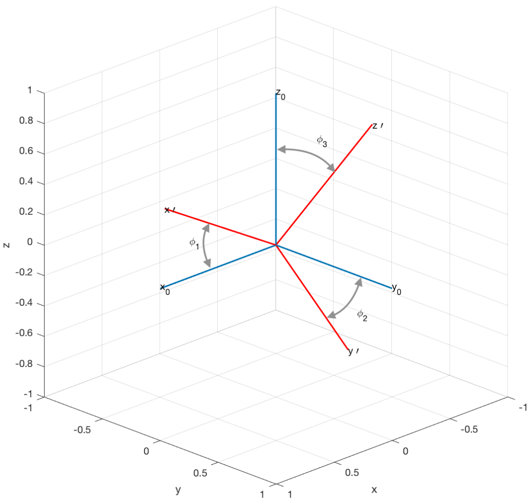

FIGURE 1: DEFINITION OF $\vec{\phi}$.

Case II

When only one angle is zero, which means $\phi_i = 0, i = 1,2$ *or* 3, the rotation axis is coincident with the ith axis of the spatial frame. For example, when given $\phi_1 = 0$, the body fixed frame can only rotate around the x axis. This will make $\phi_2 = \phi_3 = \theta$, $\theta \in (-\pi, \pi]$ and thus the rotation matrix is

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{pmatrix}. \tag{11}$$

Similarly, when $\phi_2 = 0$, the frame rotates around y axis and

$$R_y = \begin{pmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{pmatrix}. \tag{12}$$

When $\phi_3 = 0$, the rotation axis is z axis,

$$R_z = \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13}$$

It should be noted that there is no such case that two out of the three angles are both zero and the other one is not zero.

Case III

When all the $\phi_i, i = 1,2,3$ are not zero, substituting Eqn.8 and Eqn.10 back to the diagonal elements of Eqn.4, we can get the closed form expression of $\vec{n}$, the elements of which are

$$n_1 = \pm\sqrt{\frac{1+\cos\phi_1-\cos\phi_2-\cos\phi_3}{3-\cos\phi_1-\cos\phi_2-\cos\phi_3}} \tag{14}$$

$$n_2 = \pm\sqrt{\frac{1-\cos\phi_1+\cos\phi_2-\cos\phi_3}{3-\cos\phi_1-\cos\phi_2-\cos\phi_3}} \tag{15}$$

$$n_3 = \pm\sqrt{\frac{1-\cos\phi_1-\cos\phi_2+\cos\phi_3}{3-\cos\phi_1-\cos\phi_2-\cos\phi_3}}. \tag{16}$$

Note that the signs of elements in $\vec{n}$ determine the octant that $\vec{n}$ lies in. Each of the eight combinations of the signs corresponds to an octant respectively. To relate the sign to ϕ, define the step function $\sigma(x)$ as

$$\sigma(x) = \begin{cases} 1, & \text{when } x < 0 \\ 0, & \text{when } x \geq 0. \end{cases} \tag{17}$$

The relationship between $\sigma(\phi)$ and the sign of ϕ is

$$\sigma(\phi) = \frac{1-\mathrm{sgn}(\phi)}{2}. \tag{18}$$

Note that there are some repeated elements in $\vec{n}$. The denominators of element under square root for $n_i, i = 1,2,3$ are the same. To simplify the representation, define

$$\begin{aligned} p_0(\vec{\phi}) &= 3-\cos\phi_1-\cos\phi_2-\cos\phi_3 \\ p_1(\vec{\phi}) &= 1+\cos\phi_1-\cos\phi_2-\cos\phi_3 \\ p_2(\vec{\phi}) &= 1-\cos\phi_1+\cos\phi_2-\cos\phi_3 \\ p_3(\vec{\phi}) &= 1-\cos\phi_1-\cos\phi_2+\cos\phi_3 \\ p_4(\vec{\phi}) &= 1+\cos\phi_1+\cos\phi_2+\cos\phi_3. \end{aligned} \tag{19}$$

Therefore, $\vec{n}$ is

$$\vec{n} = \begin{pmatrix} (-1)^{\sigma(\phi_1)}\sqrt{\frac{p_1}{p_0}} \\ (-1)^{\sigma(\phi_2)}\sqrt{\frac{p_2}{p_0}} \\ (-1)^{\sigma(\phi_3)}\sqrt{\frac{p_3}{p_0}} \end{pmatrix}. \tag{20}$$

By substituting $\vec{n}$ and θ back to Eqn.4, we can get the rotation matrix $R = (c_1, c_2, c_3)$ in terms of ϕ, where

$$
c_1 = \begin{pmatrix} \cos\phi_1 \\ \frac{1}{2}((-1)^{\sigma(\phi_1)+\sigma(\phi_2)}\sqrt{p_1p_2} + (-1)^{\sigma(\phi_3)}\sqrt{p_3p_4}) \\ \frac{1}{2}((-1)^{\sigma(\phi_1)+\sigma(\phi_3)}\sqrt{p_1p_3} + (-1)^{1+\sigma(\phi_2)}\sqrt{p_2p_4}) \end{pmatrix}
$$
$$
c_2 = \begin{pmatrix} \frac{1}{2}((-1)^{\sigma(\phi_1)+\sigma(\phi_2)}\sqrt{p_1p_2} + (-1)^{1+\sigma(\phi_3)}\sqrt{p_3p_4}) \\ \cos\phi_2 \\ \frac{1}{2}((-1)^{\sigma(\phi_2)+\sigma(\phi_3)}\sqrt{p_2p_3} + (-1)^{\sigma(\phi_1)}\sqrt{p_1p_4}) \end{pmatrix}
$$
$$
c_3 = \begin{pmatrix} \frac{1}{2}((-1)^{\sigma(\phi_1)+\sigma(\phi_3)}\sqrt{p_1p_3} + (-1)^{\sigma(\phi_2)}\sqrt{p_2p_4}) \\ \frac{1}{2}((-1)^{\sigma(\phi_2)+\sigma(\phi_3)}\sqrt{p_2p_3} + (-1)^{1+\sigma(\phi_1)}\sqrt{p_1p_4}) \\ \cos\phi_3 \end{pmatrix}. \tag{21}
$$

For instance, when $\vec{n}$ is in Octant I, which means the sign of n_i is "+", the rotation matrix is

$$
\begin{pmatrix} \cos\phi_1 & \frac{1}{2}(\sqrt{p_1p_2} - \sqrt{p_3p_4}) & \frac{1}{2}(\sqrt{p_1p_3} + \sqrt{p_2p_4}) \\ \frac{1}{2}(\sqrt{p_1p_2} + \sqrt{p_3p_4}) & \cos\phi_2 & \frac{1}{2}(\sqrt{p_2p_3} - \sqrt{p_1p_4}) \\ \frac{1}{2}(\sqrt{p_1p_3} - \sqrt{p_2p_4}) & \frac{1}{2}\sqrt{p_2p_3} + \sqrt{p_1p_4}) & \cos\phi_3 \end{pmatrix}. \tag{22}
$$

Note that the element under square root in the elements of the rotation matrix remains the same no matter which octant that $\vec{n}$ lies in. However, the sign before the square root element varies depending on the octant. Another thing to mention is that by substituting $\phi_i = 0$ into the rotation matrix for Case III which is Eqn.21, we can get the corresponding matrix in Case II. The reason why we discuss Case II and Case III separately is that Case II is a unique type of singularity which will be further developed later in the paper.

Equivalent Demonstration of Rotation Matrix

Here we show the relationship between $\vec{\phi}$ and rotation matrix R, and the relationship between $(\vec{n}, \theta)$ and $\vec{\phi}$. Given any rotation matrix $R \in SO(3)$, we can get the unit rotation axis vector $\vec{n}$ and the magnitude θ as

$$
\vec{n} = \left(\frac{R - R^T}{2 \cdot \sin\theta} \right)^{\vee} \tag{23}
$$
$$
\theta = \arccos \frac{tr(R) - 1}{2}. \tag{24}
$$

$\vec{\phi}$ can be determined given rotation matrix based on Eqn.8 and Eqn.20. First define the revised sign function as

$$
\text{sgnrev}(x) = \begin{cases} 1, & when\ x > 0 \\ 1\ or\ -1, & when\ x = 0 \\ -1, & when\ x < 0 \end{cases}. \tag{25}
$$

Then we can have a relatively simple representation for $\vec{\phi}$ as

$$
\vec{\phi} = \begin{pmatrix} \text{sgnrev}(n_1) \cdot \arccos(r_{11}) \\ \text{sgnrev}(n_2) \cdot \arccos(r_{22}) \\ \text{sgnrev}(n_3) \cdot \arccos(r_{33}) \end{pmatrix}. \tag{26}
$$

If the rotation vector $\vec{n}$ and magnitude θ are given, one way is to refer the diagonal elements of R in terms of $(\vec{n}, \theta)$ in Eqn.4. We can have $\vec{\phi}$ as

$$
\vec{\phi} = \begin{pmatrix} \text{sgnrev}(n_1) \cdot \arccos\{c\theta - n_1^2\cos(\theta) + n_1^2\} \\ \text{sgnrev}(n_2) \cdot \arccos\{c\theta - n_2^2\cos(\theta) + n_2^2\} \\ \text{sgnrev}(n_3) \cdot \arccos\{(c\theta - 1)\left(n_1^2 + n_2^2\right) + 1\} \end{pmatrix}. \tag{27}
$$

Conversely, given $\vec{\phi}$, we can get the rotation matrix R via Eqn.21. The magnitude is derived by Eqn.10 and the rotation vector by Eqn.20.

CONFIGURATION SPACE

To find the configuration space of $\vec{\phi}$, we first concentrate on the range of $p_j, j = 0, 1, ..., 4$ defined above.

Since $R \in SO(3)$ is a rotation matrix, the elements of R have to be real numbers. We have to restrict the square roots in R to be real numbers. For Case III, p_0 is always a positive real number. Thus, $q_j, j = 1, 2, 3, 4$ should be non-negative. Together with the configuration space for Case I which is the origin in $\vec{\phi}$ space when $p_0 = 0$, the constraints imposed on configuration space can then be concluded as

$$
\begin{aligned} p_0 &\geq 0 \\ p_1 &\geq 0 \\ p_2 &\geq 0 \\ p_3 &\geq 0 \\ p_4 &\geq 0. \end{aligned} \tag{28}
$$

By plotting equations as $q_j = 0, j = 1, 2, 3, 4$ implicitly in MATLAB, we can get the surface of configuration space of $\vec{\phi}$ visualized as Fig.2. The range of ϕ_i is $[-\pi, \pi]$. The surface generated by the same equation is shown in the same color. All the

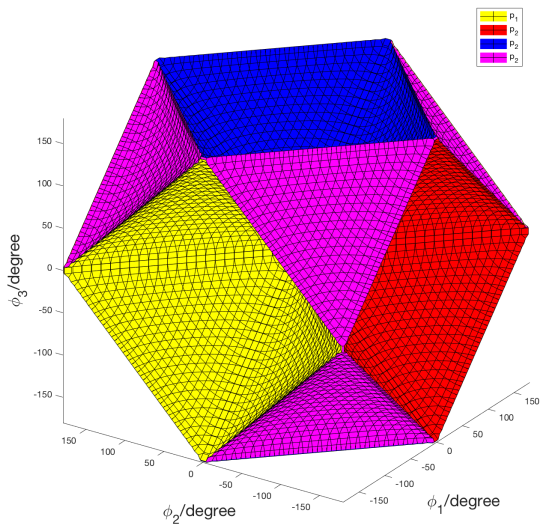

FIGURE 2: $\vec{\phi}$ SPACE.

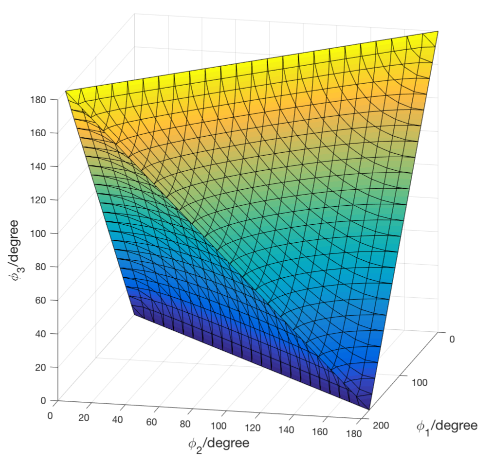

FIGURE 3: OCTANT I OF $\vec{\phi}$ SPACE.

configurations inside and on the surface form the configuration space. Note that that the configuration space is centrally symmetric about the origin. There are eight identical concave tetrahedral units and each tetrahedron sits in one octant in $\vec{\phi}$ space. For example, the tetrahedron in Octant I is shown in Fig. 3. Concave tetrahedron means that the surface of tetrahedron is concave.

Relationship between Configuration Spaces

It is known that one geometric model of the rotation group $SO(3)$ is a solid ball of radius π. Each dot in the ball represents a rotation configuration with rotation vector from the origin to the point and magnitude as the distance of the point to the origin. The issue left is that the rotation through π and $-\pi$ are the same. In other words, the points on the hypersphere of the ball that are symmetric about the origin are connected. The solid ball could also be seen as the configuration space for the axis-angle coordinates.

Knowing the configuration space of $\vec{\phi}$, we can construct a surjective(onto) function from $\vec{\phi}$ space to the solid ball. More precisely, the function can map each octant of $\vec{\phi}$ space to 1/8 of the solid ball that lies in the corresponding octant. A similar issue also happens: the points on the surface when $\phi_i = \pi$ is connected with the symmetric points about the origin which satisfies $\phi_i = -\pi$.

Though the two geometric models look quite different, the $\vec{\phi}$ space can be regarded as a reshaped model of the ball. The relationship of surface p_1, p_2 and p_3 in $\vec{\phi}$ space and the octant plane in $(\theta,\vec{n})$ space is shown in Tab.1. The mapping between p_4 in $\vec{\phi}$ space and sphere in $(\theta,\vec{n})$ space is shown in Tab.2. Those two tables follow similar convention. The first column shows the type of surface in $\vec{\phi}$ space and the second column is the octant that the surface lies in. The third and forth column is the corresponding configuration in $(\theta,\vec{n})$ space. For example, $n_2 - n_3$ in Tab.1 is the plane determined by n_2 and n_3 axis. The forth column shows the correspond quadrant. For Tab.2, the third column denotes the sphere and the forth is the octant that the corresponding surface lies in. In Tab.1, the two octants in the second column means that they can both be mapped to the same surface in $(\theta,\vec{n})$ space and thus they are actually connected in $\vec{\phi}$ space. This property will be used in the discussion of application.

PROPERTIES OF ROTATION MATRICES

Inverse and Transpose of a Rotation

By direct calculation, the inverse as well as the transpose of the rotation $R(\vec{\phi})$ is just the rotation matrix in terms of $-\vec{\phi}$.

$$R(\vec{\phi})^{-1} = R(\vec{\phi})^T = R(-\vec{\phi}). \tag{29}$$

Metrics for Rotations

Metric Candidates Metric can be seen as the distance in configuration space and could be used in various areas such as motion planning [18]. A metric d for $SO(3)$ group should satisfy usual axioms [19] for metrics which are

$$\begin{aligned} &d(R_1,R_2) = 0 \Leftrightarrow R_1 = R_2 \\ &d(R_1,R_2) = d(R_2,R_1)\ \forall R_1,R_2 \in SO(3) \\ &d(R_1,R_3) \le d(R_1,R_2) + d(R_2,R_3)\ \forall R_1,R_2,R_3 \in SO(3). \end{aligned} \tag{30}$$

TABLE 1: MAPPING OF SURFACE FOR p_1, p_2 AND p_3.

$\vec{\phi}$ space	Octant	$(\theta, \vec{n})$ space	Quadrant
p_1	I or II	$n_2 - n_3$	I
p_1	III or IV	$n_2 - n_3$	II
p_1	VII or VIII	$n_2 - n_3$	III
p_1	V or VI	$n_2 - n_3$	IV
p_2	I or IV	$n_1 - n_3$	I
p_2	II or III	$n_1 - n_3$	II
p_2	VI or VII	$n_1 - n_3$	III
p_2	V or VIII	$n_1 - n_3$	IV
p_3	I or V	$n_1 - n_2$	I
p_3	II or VI	$n_1 - n_2$	II
p_3	III or VII	$n_1 - n_2$	III
p_3	IV or VIII	$n_1 - n_2$	IV

TABLE 2: MAPPING OF SURFACE FOR p_4.

$\vec{\phi}$ space	Octant	$(\theta, \vec{n})$ space	Octant
p_4	I	Sphere	I
p_4	II	Sphere	II
p_4	III	Sphere	III
p_4	IV	Sphere	IV
p_4	V	Sphere	V
p_4	VI	Sphere	VI
p_4	VII	Sphere	VII
p_4	VIII	Sphere	VIII

where $R_1, R_2 \in SO(3)$.

Another useful property is bi-invariance, which means that for every R_0 in $SO(3)$:

$$d(R_0R_1, R_0R_2) = d(R_1R_0, R_2R_0) = d(R_1, R_2). \tag{31}$$

We first define the map from $\vec{\phi}$ to rotation matrix R as f, $f : \vec{\phi} \to R$. The inverse operation can be denoted as $f^{-1} : R \to \vec{\phi}$.

The relative ϕ between the two body fixed frames rotated via R_1 and R_2 respectively is

$$\vec{\phi} = f^{-1}(R_1^T R_2), \tag{32}$$

the elements of which is

$$\phi_i = \text{signrev}(n_i)\cos^{-1}((R_1^T R_2)_{ii}). \tag{33}$$

Note that the absolute value of ϕ_i is the angle between the ith axes of two body frames rotated by R_1 and R_2. This angle is still measured in Euclidean space.

It is known that rotation magnitude θ in Rodrigues formula is a well defined metric for $SO(3)$. Thus, by representing θ by $\vec{\phi}$, we can get a good metric as

$$d(R_1, R_2) = \arccos \frac{\cos\phi_1 + \cos\phi_2 + \cos\phi_3 - 1}{2}. \tag{34}$$

Other metric candidates such as

$$d_1(R_1, R_2) = |\phi_1| + |\phi_2| + |\phi_3| \tag{35}$$

$$d_2(R_1, R_2) = \sqrt{(\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2} \tag{36}$$

$$d_3(R_1, R_2) = \max_i |\phi_i|. \tag{37}$$

are all left-invariant but not right-invariant.

Invariant Property The proof of the left-invariant property is as follows.

Proof. When multiplying the same rotation matrix R_0 on the left side, the relative ϕ changes to

$$\begin{aligned}\vec{\phi}_L &= f^{-1}((R_0R_1)^T(R_0R_2)) = f^{-1}((R_1^T R_0^T)(R_0R_2)) \\ &= f^{-1}(R_1^T(R_0^T R_0)R_2) = f^{-1}(R_1^T R_2) = \vec{\phi}.\end{aligned}$$

Since the result $\vec{\phi}_L$ is the same as the initial $\vec{\phi}$, the metrics derived from $\vec{\phi}_L$ and $\vec{\phi}$ should also be the same, which means the three metric candidates are all left-invariant.

Similar derivation can also be applied on the case when right multiplying R_0.

$$\begin{aligned}\vec{\phi}_R &= f^{-1}((R_1R_0)^T(R_2R_0)) = f^{-1}((R_0^T R_1^T)(R_2R_0)) \\ &= f^{-1}(R_0^T(R_1^T R_2)R_0),\end{aligned}$$

TABLE 3: EXAMPLE OF METRICS

Metric	d_1	d_2	d_3
(R_1R_0, R_2R_0)	4.0397	2.3680	1.6744
(R_0R_1, R_0R_1)	4.0472	2.3502	1.5073
(R_1, R_2)	4.0472	2.3502	1.5073

which is not the same as $\vec{\phi}$ unless R_0, R_1 and R_2 commute. Multiplication of rotation matrices is usually not commutative except for considering small angles or rotations around the same axis. However, the value of $\vec{\phi}$ being different can not be regarded as a sufficient condition for the nonexistence of the right-invariant property, because it is possible for $R(\vec{\phi}_1) = R(\vec{\phi}_2)$ even if $\vec{\phi}_1 \neq \vec{\phi}_2$.

To prove the metrics are not right-invariant, we can give a counterexample. Suppose $\vec{\phi}_1 = [2; 1.3; 1.5]$, $\vec{\phi}_2 = [-1.7; 2; 1.5]$, $\vec{\phi}_0 = [1.6; 0.7; 1.5]$. The numerical values of metrics are listed in Tab. 3. By comparing the first and third row, we can tell the three metric candidates are not right-invariant.

Triangle Property Another property which is interesting to prove is the triangle inequality shown in Eqn.30. The proofs are as follows.

Proof. For d_1, triangle inequality could be expanded and recombined as

$$\begin{aligned} & d_1(R_1,R_2) + d_1(R_2,R_3) - d_1(R_1,R_3) \\ =& |\phi_1^{12}| + |\phi_2^{12}| + |\phi_3^{12}| + |\phi_1^{23}| + |\phi_2^{23}| + |\phi_3^{23}| - |\phi_1^{13}| - |\phi_2^{13}| - |\phi_3^{13}| \\ =& (|\phi_1^{12}| + |\phi_1^{23}| - |\phi_1^{13}|) + (|\phi_2^{12}| + |\phi_2^{23}| - |\phi_2^{13}|) \\ & + (|\phi_2^{12}| + |\phi_2^{23}| - |\phi_2^{13}|) \end{aligned} \tag{38}$$

where the subscript of ϕ symbols the axis. The superscript represents the two relative frames.

To prove the property, we use Cauchy-Schwarz inequality for angles between vectors [20]

$$\psi_{xz} \leq \psi_{xy} + \psi_{yz} \tag{39}$$

where $x, y, z \in \mathbb{C}^3$ are three vectors and ψ is the angle between them. Thus we could get that each bracket in Eqn.38 is greater or equal to 0. It shows that d_1 satisfies the triangle inequality.

Proof. For d_2, to simplify the representation, we use x to replace ϕ^{13}, y for ϕ^{12} and z for ϕ^{23}. The triangle property that we seek is

$$\sqrt{x_1^2 + x_2^2 + x_3^2} \leq \sqrt{y_1^2 + y_2^2 + y_3^2} + \sqrt{z_1^2 + z_2^2 + z_3^2}. \tag{40}$$

By taking the square of Eqn.40 on both sides, we have

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 \leq & \; y_1^2 + y_2^2 + y_3^2 + z_1^2 + z_2^2 + z_3^2 + \\ & 2\sqrt{(y_1^2 + y_2^2 + y_3^2)(z_1^2 + z_2^2 + z_3^2)}. \end{aligned} \tag{41}$$

Using Cauchy-Schwarz inequality again, we have

$$x_1^2 + x_2^2 + x_3^2 \leq y_1^2 + z_1^2 + 2y_1z_1 + y_2^2 + z_2^2 + 2y_2z_2 + x_3^2 y_3^2 + z_3^2 + 2y_3z_3. \tag{42}$$

Comparing Eqn.41 and Eqn.42, we could notice that if

$$y_1z_1 + y_2z_2 + y_3z_3 \leq \sqrt{(y_1^2 + y_2^2 + y_3^2)(z_1^2 + z_2^2 + z_3^2)}. \tag{43}$$

holds, Eqn.40 holds. Taking square of Eqn.43 and subtracting right hand side from left hand side, we have

$$\begin{aligned} & (y_1z_1 + y_2z_2 + y_3z_3)^2 - (y_1^2 + y_2^2 + y_3^2)(z_1^2 + z_2^2 + z_3^2) \\ =& -(y_1z_2 - y_2z_1)^2 - (y_1z_3 - y_3z_1)^2 - (y_2z_3 - y_3z_2)^2 \leq 0. \end{aligned} \tag{44}$$

Thus Eqn.40 holds. Metric d_2 satisfies triangle property.

Proof. For d_3, suppose ϕ_1^{13} is the maximum value among $\vec{\phi}^{13}$. From Krein's inequality, we have

$$\phi_1^{13} \leq \phi_1^{12} + \phi_1^{23}. \tag{45}$$

Since $\phi_1^{12} \leq \max \vec{\phi}^{12}$ and $\phi_1^{23} \leq \max \vec{\phi}^{23}$, the triangle inequality for metric d_3 holds.

JACOBIAN AND SINGULARITY ANALYSIS

Usually, the singular configuration is where the Jacobian matrix loses its usual rank, i.e $\det(J) = 0$. When only using $\vec{\phi}$ which contains three coordinates to represent the rotation in $SO(3)$, singularity might be artificially induced by the angles' definition. Jacobian matrix could map the velocity in configuration space to Euclidean space.

Spatial and Body Jacobian of a Rotation

Based on the chain rule, the partial derivative of $R(\vec{\phi})$ is

$$\dot{R} = \frac{\partial R}{\partial \phi_1}\dot{\phi}_1 + \frac{\partial R}{\partial \phi_2}\dot{\phi}_2 + \frac{\partial R}{\partial \phi_3}\dot{\phi}_3. \tag{46}$$

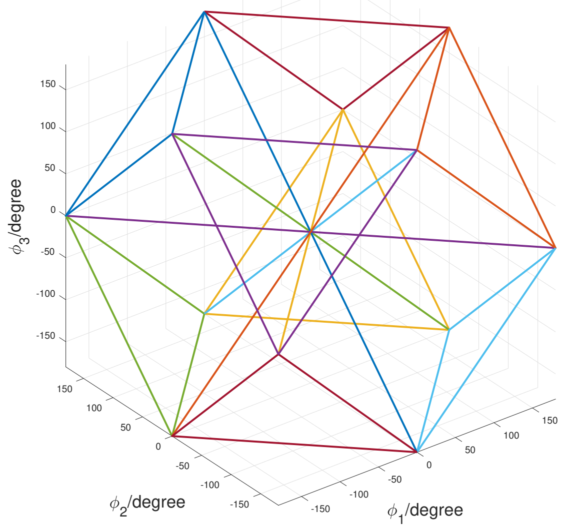

FIGURE 4: SWITCHING SINGULARITIES.

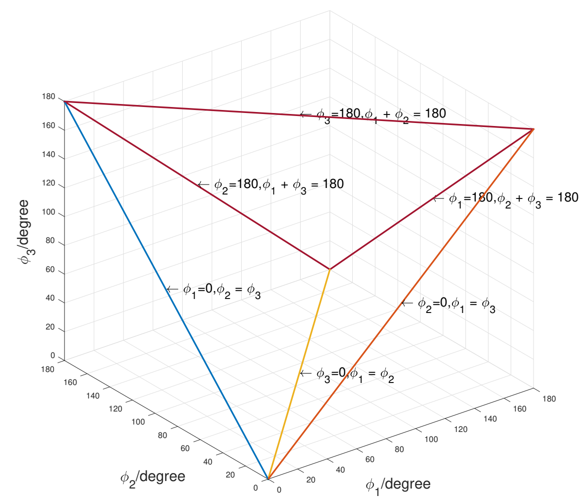

FIGURE 5: SWITCHING SINGULARITIES IN OCTANT I.

The body Jacobian matrix is

$$J_b(\phi_1,\phi_2,\phi_3) = \left[\left(R^T\frac{\partial R}{\partial \phi_1}\right)^\vee, \left(R^T\frac{\partial R}{\partial \phi_2}\right)^\vee, \left(R^T\frac{\partial R}{\partial \phi_3}\right)^\vee\right]. \tag{47}$$

The spatial Jacobian matrix is

$$J_s(\phi_1,\phi_2,\phi_3) = \left[\left(\frac{\partial R}{\partial \phi_1}R^T\right)^\vee, \left(\frac{\partial R}{\partial \phi_2}R^T\right)^\vee, \left(\frac{\partial R}{\partial \phi_3}R^T\right)^\vee\right]. \tag{48}$$

Knowing that Eqn. 21 is a general version of rotation matrix suitable for Case II and Case III, we can get the body and spatial Jacobian matrix for it. Although the closed form of Jacobian matrix looks big, the determinants of spatial and body Jacobian are the same. For example, the Jacobian Matrix in Octant I is $J_b = (Jc_1, Jc_2, Ic_3)$, where

$$Jc_1 = \frac{1}{2p_0\sqrt{p_1p_2p_3p_4}} \cdot \tag{49}$$
$$\begin{pmatrix} -p_0(c\phi_2+c\phi_3)s\phi_1\sqrt{p_2p_3} - p_0\sqrt{p_1p_4}(c\phi_2-c\phi_3)s\phi_1 \\ \sqrt{p_1p_3}(p_1+p_0c\phi_2+2)s\phi_1 - p_0\sqrt{p_2p_4}(c\phi_2-1)s\phi_1 \\ (-p_3+p_0c\phi_3+4)s\phi_1\sqrt{p_1p_2} + p_0(c\phi_3-1)s\phi_1\sqrt{p_3p_4} \end{pmatrix}$$

$$Jc_2 = \frac{1}{2p_0\sqrt{p_1p_2p_3p_4}} \cdot \tag{50}$$
$$\begin{pmatrix} p_0(c\phi_1+1)s\phi_2\sqrt{p_2p_3} + p_0(c\phi_1-1)s\phi_2\sqrt{p_1p_4} \\ p_0\sqrt{p_2p_4}(c\phi_1-c\phi_3)s\phi_2 - p_0\sqrt{p_1p_3}(c\phi_1+c\phi_3)s\phi_2 \\ \sqrt{p_1p_2}(-p_3+p_0c\phi_3+4)s\phi_2 - p_0\sqrt{p_3p_4}(c\phi_3-1)s\phi_2 \end{pmatrix}$$

$$Jc_3 = \frac{1}{2p_0\sqrt{p_1p_2p_3p_4}} \cdot \tag{51}$$
$$\begin{pmatrix} p_0\sqrt{p_2p_3}(c\phi_1+1)s\phi_3 - p_0\sqrt{p_1p_4}(c\phi_1-1)s\phi_3 \\ (p_1+p_0c\phi_2+2)s\phi_3\sqrt{p_1p_3} + p_0(c\phi_2-1)s\phi_3\sqrt{p_2p_4} \\ -p_0(c\phi_1+c\phi_2)s\phi_3\sqrt{p_1p_2} - p_0\sqrt{p_3p_4}(c\phi_1-c\phi_2)s\phi_3 \end{pmatrix}.$$

Also, although the rotation matrices in each octant are different, the absolute values of determinant of Jacobian are the same as Eqn.52.

$$|J| = \frac{|s\phi_1 s\phi_2 s\phi_3|}{\sqrt{p_1p_2p_3p_4}}. \tag{52}$$

Switching Singularities

A switching singularity is defined to be when the numerator of Jacobian is zero. It happens when the configuration switches between different cases discussed in forward kinematics part.

The singular configuration is shown in Fig.4. The analytical expression for singular configuration in Octant I is shown in Fig.5. This kind of singularity contains three types.

The first kind of switching singularity corresponds to configurations when

$$\phi_i = 0, i = 1,\ 2,\ or\ 3.$$

In this case, a singular rotation configuration satisfies both Case II and Case III. In other words, a rotation configuration can be seen as a switching point between Case III and II. In Case II, body fixed frame rotate around one of its axes. So there is an extra constraint that $\phi_j = \phi_k, j \neq k \neq i$. This will cause the loss of degrees of freedom.

The origin, where all of the three angles are 0 which is Case I, is also a singular configuration.

The third case is when

$$\phi_i = \pi\ or - \pi,\ i = 1,\ 2\ or\ 3.$$

For example, when $i = 1$, the body fixed frame rotates around the $-x$ axis. This could be seen as identical to Case II since rotating around x by θ is the same as rotating around $-x$ by $-\theta$. The extra constraint that $\phi_j + \phi_k = \pi$ will also result in the DOF's deficiency.

Boundary Singularities A boundary singularity is defined to be when the denominator of Jacobian is zero. Based on Eqn.52, when each $p_j = 0, j = 1,2,3\ or\ 4$ equals to 0, there is a boundary singular configuration. Thus, all the configurations on surface of the $\vec{\phi}$ space except for the switching singular configurations are boundary singularities. It means very small ϕ space velocity can be converted into quite large angular velocity. The body fixed frame would be out of control.

APPLICATION

One important thing to note is that the configuration coordinate for three direction cosine coordinates is $\vec{\phi}$ instead of $\cos\vec{\phi} = (\cos\phi_1, \cos\phi_2, \cos\phi_3)$. Based on the rotation matrix in Eqn.21, we can tell that the expression is relative to the sign of $\vec{\phi}$ which could not be recovered from the cosine value. Thus, when given a single point in $\cos\vec{\phi}$ space, it is not possible to determine the octant that $\vec{\phi}$ lies in and therefore cannot get the unique rotation matrix.

However, in real rigid body motion, when trying to control the body-fixed frame from an initial orientation to a final orientation via $(\cos\vec{\phi})(t)$, we could use the continuity of trajectory to determine the signs of $\vec{\phi}$. The continuity of trajectory means the continuity of both $\vec{\phi}$ and $\dot{\vec{\phi}}$. For $\vec{\phi}$, the configuration space in each octant are connected not only at the origin, but also on the surface. For example, the point $Q_1 = (q_1, q_2, q_3)$ on p_1 surface in Octant I and point $Q_2 = (-q_1, q_2, q_3)$ on p_1 surface in Octant II are connected. More information about connected surface can be found in the first two columns in Tab.1.

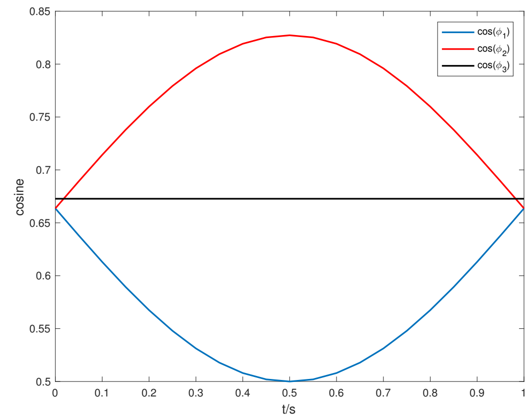

FIGURE 6: $(\cos\vec{\phi})(t)$.

Here we give an example of a trajectory $(\cos\vec{\phi})(t)$ as shown in Fig.6. The whole time length is one second. The blue, red and black lines are $\cos\phi_1(t)$, $\cos\phi_2(t)$ and $\cos\phi_3(t)$, respectively. The initial point in $\vec{\phi}$ space is given as $\phi_0 = (0.8451, 0.8451, 0.8329)$ in Octant I. The final configuration is $\phi_f = (-0.8451, 0.8451, 0.8329)$ in Octant II.

Note that the equilibrium points along $\cos\phi_i(t)$ is when the point in $\vec{\phi}$ space reaches the boundary of the $\vec{\phi}$ space. Since there is only one equilibrium for $\cos\phi_i$, to enter Octant II, $\dot{\vec{\phi}}$ should point away from Octant I at that moment. So that the point will then get out of Octant I and enter Octant II at the next time step. This will cause the change of sign for ϕ_1. The black line in Fig.7 shows the trajectory of $\vec{\phi}(t)$. The black dotted line denotes the sign change.

CONCLUSION

In this paper, we proposed a new parameterization method to represent rotation matrix using the $\vec{\phi}$ angles recoverd from the three direction cosines that lies on the diagonal part. We find the configuration space in terms of $\vec{\phi}$. It is centrally symmetric with the origin. An onto map from $\vec{\phi}$ space to the solid ball for $SO(3)$ is constructed which is useful to understand the singularities. We also find a bi-invariant metric and two left-invariant metric for measuring the distance in configuration space which could be the foundation for path planning in $\vec{\phi}$ space.

An advantage of the new parameterization presented in this

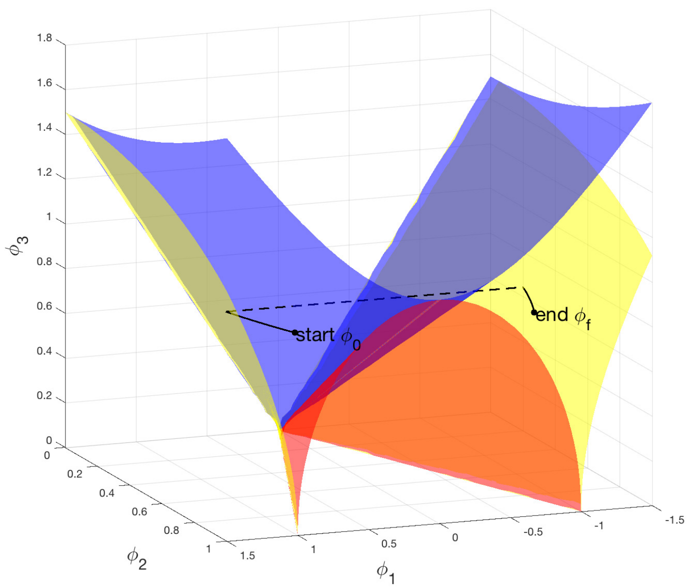

FIGURE 7: $\vec{\phi}(t)$.

paper is the intuitive nature of the variables, which are the angles between the spatial fixed frame and the body fixed frame. One negative feature of this parameterization is the singularities induced by the definition of $\vec{\phi}$. This potentially can be solved by introducing an extra angle into the variables, such as using a non-orthogonal frame with four axes instead of three. Modifying the current formulation to represent rotation matrices with more angles in non-orthogonal frames will be explored in the future.

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